

# Quillen-Suslin theory for a structure theorem for the Elementary Symplectic Group

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**Abstract:** A new set of elementary symplectic elements is described, It is shown that these also generate the elementary symplectic group  $\mathrm{ESp}_{2n}(R)$ . These generators are more symmetrical than the usual ones, and are useful to study the action of the elementary symplectic group on unimodular rows. Also, an alternate proof of,  $\mathrm{ESp}_{2n}(R)$  is a normal subgroup of  $\mathrm{Sp}_{2n}(R)$ , is shown using the Local Global Principle of D. Quillen for the new set of generators.

## 1 Introduction

Let  $R$  be a commutative ring with 1. The symplectic group  $\mathrm{Sp}_{2n}(R)$  is the isotropy group of  $\psi_n$  under the action of  $\mathrm{SL}_{2n}(R)$  on  $\psi_n$  by conjugation, i.e.  $\mathrm{Sp}_{2n}(R) = \{\alpha \in \mathrm{SL}_{2n}(R) \mid \alpha^t \psi_n \alpha = \psi_n\}$ . (Here  $\psi_n \in \mathrm{SL}_{2n}(\mathbb{Z})$  denotes the usual alternating matrix (of Pfaffian one) got by placing the standard  $2 \times 2$  alternating matrix  $\psi_1$  of Pfaffian one, diagonally  $n$  times.

The Elementary Group  $E_n(R)$  is the subgroup of  $\mathrm{GL}_n(R)$  generated by  $\{E_{ij}(\lambda) : \lambda \in R\}$ , for  $i \neq j$ , where  $E_{ij}(\lambda) = I_n + \lambda e_{ij}$  and  $e_{ij}$  is the matrix with 1 on the  $(i, j)$ -th position and 0 elsewhere.  $I_n$  denotes the identity matrix.

The Elementary Symplectic group  $\mathrm{ESp}_{2n}(R)$  is the subgroup of  $\mathrm{Sp}_{2n}(R)$  generated by the following ‘‘symplectic elementary’’ matrices  $E_{21}(x), E_{12}(x)$ , for  $x \in R$ ,  $S_{ij}(\lambda)$ ,  $1 \leq i \neq j \neq \pi(i) \leq 2n$ ,  $\lambda \in R$ , where  $\pi$  is the permutation  $(1\ 2)(3\ 4) \cdots (2n-1\ 2n) \in S_{2n}$ :

$$S_{ij}(\lambda) = I_{2n} + \lambda e_{ij} - (-1)^{i+j} \lambda e_{\pi(j)\pi(i)}.$$

In this note we describe a more ‘‘symmetric’’ set of generators denoted by  $E(A)$ ,  $E(B)$ ,  $E(C)$ ,  $E(D)$ , which generate  $\mathrm{ESp}_{2n}(R)$ .

These generators are useful in analysing the action of the elementary group on a unimodular row. A recent result in [3], which generalizes a famous lemma of L.N. Vaserstein in ([12], Lemma 5.6), states that the elementary orbit of a unimodular row coincides with its elementary symplectic orbit. L.N. Vaserstein used this result in [12] to prove that the orbit space of unimodular rows of length three modulo elementary action is isomorphic to the elementary symplectic Witt

group  $W_E(R)$  over a two dimensional ring  $R$ . Such an isomorphism was also shown for a non-singular affine algebra of dimension three over an algebraically closed field  $k$ , when characteristic  $k \neq 2, 3$  by Rao-van der Kallen in [7].

The theorem in this note enables one to study elementary symplectic action, in a more symmetric way. The objective of the second named author was to consider if the Vaserstein symbol from the orbit space  $Um_3(A)/E_3(A) \rightarrow W_E(A)$  is injective, when  $A$  is a three dimensional affine algebra over an algebraically closed field (even if  $A$  is singular). This leads to the study of the action of a 1-stably elementary matrix on an alternating matrix; which is discussed in [4]. It is the hope of the second author that the structure theorem developed here will be useful to analyse this.

**Convention:** In this article, we assume  $R$  is a commutative ring with 1 and 2 is an invertible element in the ring  $R$ .

## 2 The initial structure theorem

To describe the initial structure theorem we isolate the following four types of basic elementary symplectic generators  $E(A)$ ,  $E(B)$ ,  $E(C)$ ,  $E(D)$  respectively which we will use in the sequel. We begin with a notation.

**Notation 2.1.** We denote by

$$E \begin{pmatrix} \lambda a & \lambda b \\ \mu a & \mu b \end{pmatrix} : = \begin{pmatrix} I_2 & X \\ \psi_{n-1} X^t \psi_1 & I_{2n-2} \end{pmatrix},$$

for some  $X \in M_{2,2n-2}(R)$  of the form  $X = (X_3, X_5, \dots, X_{2n-1})$ , where

$$X_{2i-1} = \begin{pmatrix} \lambda a_i & \lambda b_i \\ \mu a_i & \mu b_i \end{pmatrix},$$

for some  $\lambda, \mu, a_i, b_i \in R$ , for  $2 \leq i \leq n$ . This is a symplectic matrix since  $\det(X_i) = 0$ , for all  $i$ .

We can also write this as  $E((X_3)_2, (X_5)_3 \dots (X_{2n-1})_n)$  where the lower indices indicate that the block  $X_{2i-1}$  lies in the  $(1, i)$ -th position of the block matrix  $E \begin{pmatrix} \lambda a & \lambda b \\ \mu a & \mu b \end{pmatrix} \in Sp_n(M_2(R))$ .

We will denote by  $E^k \begin{pmatrix} \lambda a & \lambda b \\ \mu a & \mu b \end{pmatrix}$ , a matrix of the above type which has **precisely** the  $X_{2k-1}$  block that is non-zero. (The  $k$  will usually be clear from the context). Note also that this is the same as  $E(X_k)$ .

For  $\alpha \in SL_r(R)$ ,  $\beta \in SL_s(R)$ ,  $\alpha \perp \beta$  is the matrix  $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in SL_{r+s}(R)$ .

Examples are stated below in the form of a lemma.

**Lemma 2.2.** For  $n > 1$ , we have

$$E_{21}(\lambda) S_{12n-1}(x) S_{12n}(y) E_{21}(-\lambda) = (\delta \perp I_{2n-2}) E^n \begin{pmatrix} x & y \\ \lambda x & \lambda y \end{pmatrix},$$

for  $\delta = E_{21}(\lambda) E_{12}(xy) E_{21}(-\lambda) \in E_2(R)$ . □

**Lemma 2.3.** A symplectic matrix  $E \begin{pmatrix} \lambda x & \lambda y \\ \mu x & \mu y \end{pmatrix}$  is elementary symplectic if  $(\lambda, \mu) \sim_{E_2(R)} (1, 0)$ .

Proof: Let  $X_{2k-1} \neq 0$  in the given matrix, and let  $(\lambda, \mu)\varepsilon^{-1^t} = (1, 0)$ . Then

$$E \begin{pmatrix} \lambda x & \lambda y \\ \mu x & \mu y \end{pmatrix} = (\varepsilon \perp I_{2n-2}) E_{12}(-xy) S_{12k-1}(x) S_{12k}(y) (\varepsilon \perp I_{2n-2})^{-1}.$$

□

The four basic elementary symplectic matrices are

$$E \begin{pmatrix} a & a \\ a & a \end{pmatrix}, E \begin{pmatrix} b & -b \\ b & -b \end{pmatrix}, E \begin{pmatrix} c & c \\ -c & -c \end{pmatrix}, E \begin{pmatrix} -d & d \\ d & -d \end{pmatrix}.$$

We will denote these by type  $E(A(a))$ ,  $E(B(b))$ ,  $E(C(c))$ ,  $E(D(d))$  respectively; or just by  $E(A)$ ,  $E(B)$ ,  $E(C)$ ,  $E(D)$  when we are not too concerned about the actual entries, but only interested in the shape and form under discussion.

**Definition of H:**  $H(= H(R))$  is the subgroup of  $\text{ESp}_{2n}(R)$  generated by the elements of the type  $E(A)$ ,  $E(B)$ ,  $E(C)$ ,  $E(D)$ .

We shall also refer the  $2 \times 2$  matrices

$$\begin{pmatrix} a & a \\ a & a \end{pmatrix}, \begin{pmatrix} b & -b \\ b & -b \end{pmatrix}, \begin{pmatrix} c & c \\ -c & -c \end{pmatrix}, \begin{pmatrix} -d & d \\ d & -d \end{pmatrix},$$

to be of type  $A(a)$ ,  $B(b)$ ,  $C(c)$ ,  $D(d)$  respectively; or just  $A$ ,  $B$ ,  $C$ ,  $D$  to indicate the shape.

For future reference we may use the relations:

**Lemma 2.4.** Let  $x, y \in R$ . Then

$$\begin{aligned} A(x)A(y) &= A(2xy), & A(x)B(y) &= B(2xy). \\ A(x)C(y) &= 0, & A(x)D(y) &= 0. \\ B(x)A(y) &= 0, & B(x)B(y) &= 0. \\ B(x)C(y) &= A(2xy), & B(x)D(y) &= B(-2xy). \\ C(x)A(y) &= C(2xy), & C(x)B(y) &= D(-2xy). \\ C(x)C(y) &= 0, & C(x)D(y) &= 0. \\ D(x)A(y) &= 0, & D(x)B(y) &= 0. \\ D(x)C(y) &= C(-2xy), & D(x)D(y) &= D(-2xy). \end{aligned}$$

□

We will find it convenient to denote type  $E(A)$  as  $\begin{pmatrix} A(a) \\ D(a) \end{pmatrix}$ , type  $E(B)$  as  $\begin{pmatrix} B(b) \\ B(b) \end{pmatrix}$ , type  $E(C)$  as  $\begin{pmatrix} C(c) \\ C(c) \end{pmatrix}$ , type  $E(D)$  as  $\begin{pmatrix} D(d) \\ A(d) \end{pmatrix}$  as it allows us to keep in focus the shape of the “horizontal” and “vertical” components of a basic elementary symplectic matrix. Or else think of them as the “top” and “bottom” component as in the case when  $n = 2$ .

One can go further to use this notation: the reader will understand if we write for  $X, Y \in M_2(R)$ ,  $\begin{pmatrix} X \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 \\ X \end{pmatrix}$ , and also  $\begin{pmatrix} X \\ Y \end{pmatrix}$ , with some indication for the placement of the blocks  $X, Y$ . In particular, one could have  $\begin{pmatrix} B \\ B \end{pmatrix}$  with the transpose of the top  $B$  and the bottom  $B$  not being the same matrix and only being matrices having the same form. Note that in the extended notation some of the matrices are obviously not elementary symplectic of course. All this is just to say that the special elementary symplectic matrices have the “*splitting property*”:

**Lemma 2.5.** (Splitting property) *For  $X \in M_2(R)$ , with  $\det(X) = 0$ ,*

$$E(X) = \begin{pmatrix} X & \\ \psi_1 X^t \psi_1 & \end{pmatrix} = \begin{pmatrix} X \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ \psi_1 X^t \psi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \psi_1 X^t \psi_1 \end{pmatrix} \begin{pmatrix} X \\ 0 \end{pmatrix}.$$

□

We will henceforth use the notations, for  $\delta \in SL_2(R)$ ,  $I_2 + X \in E_2(R)$ ,  $X = B$  or  $C$ .  $\sigma \in SL_{2n}(R)$ , etc.

$$\begin{aligned} \delta \sigma &= (\delta \perp I_{2n-2})\sigma, \\ \sigma \delta &= \sigma(\delta \perp I_{2n-2}), \\ \sigma_{I_2+X} &= \sigma(I_{2k-2} \perp (I_2 + X) \perp I_{2n-2k}), \end{aligned}$$

for some  $k$ , which will be clear from the context.

**Lemma 2.6.** *For  $\delta \in SL_2(R)$ ,  $y_i \in R$ ,  $3 \leq i \leq 2n$ ,*

$$\delta \left( \prod_{i=3}^{2n} S_{1i}(y_i) \right) \delta^{-1} = \sigma \prod_{i=2}^n E^i \begin{pmatrix} \lambda y_{2i-1} & \lambda y_{2i} \\ \mu y_{2i-1} & \mu y_{2i} \end{pmatrix},$$

where  $(\lambda, \mu) = e_1 \delta^t$ ,  $\sigma = \delta E_{12}(y_3 y_4 + y_5 y_6 + \cdots + y_{2n-1} y_{2n}) \delta^{-1}$ .

□

**Lemma 2.7.** *Let  $R$  be a commutative ring. Then one has the identities:*

$$E \begin{pmatrix} \lambda x & \lambda y \\ \mu x & \mu y \end{pmatrix} = {}^{Ch} E \begin{pmatrix} \lambda a & \lambda a \\ \mu a & \mu a \end{pmatrix} E \begin{pmatrix} \lambda b & -\lambda b \\ \mu b & -\mu b \end{pmatrix}, \quad (1)$$

with  $a = \frac{x+y}{2}$ ,  $b = \frac{x-y}{2}$ , and where  $Ch$  is the matrix

$$\begin{aligned} Ch &= \begin{pmatrix} 1 - 2\lambda\mu ab & 2\lambda^2 ab \\ -2\mu^2 ab & 1 + 2\lambda\mu ab \end{pmatrix} \in SL_2(R) \cap E_3(R). \\ &= \varepsilon(I_2 + 2abe_1^t e_2) \varepsilon^{-1} \in SL_2(R), \end{aligned}$$

if  $(\lambda, \mu) = e_1 \varepsilon^t$ , for some  $\varepsilon \in SL_2(R)$ . (Note that  $Ch \in E_2(R)$  if  $\varepsilon \in E_2(R)$ .)

Proof: The first identity is easily verified and the second one follows immediately from it.

□

**Lemma 2.8.** *Let  $R$  be a commutative ring. One has the identities, for  $a, b, \lambda, \mu, x, y \in R$ ,*

$$E \begin{pmatrix} \lambda a & \lambda a \\ \mu a & \mu a \end{pmatrix} = E \begin{pmatrix} x & x \\ x & x \end{pmatrix} E \begin{pmatrix} y & y \\ -y & -y \end{pmatrix}_{I_2+C}, \quad (2)$$

where  $x = \frac{\lambda a + \mu a}{2}$ ,  $y = \frac{\lambda a - \mu a}{2}$ , and where

$$\begin{aligned} I_2 + C &= \begin{pmatrix} 1 + 2xy & 2xy \\ -2xy & 1 - 2xy \end{pmatrix} \in E_2(R). \\ E \begin{pmatrix} \lambda b & -\lambda b \\ \mu b & -\mu b \end{pmatrix} &= E \begin{pmatrix} x & -x \\ x & -x \end{pmatrix} E \begin{pmatrix} -y & y \\ y & -y \end{pmatrix}_{I_2+B}, \end{aligned} \quad (3)$$

where  $x = \frac{\lambda b + \mu b}{2}$ ,  $y = \frac{\mu b - \lambda b}{2}$ , and where

$$I_2 + B = \begin{pmatrix} 1 + 2xy & -2xy \\ 2xy & 1 - 2xy \end{pmatrix} \in E_2(R).$$

*Proof.* This is a direct verification.  $\square$

**Lemma 2.9.** *One has an expression of the form, for  $\lambda, \mu, x, y \in R$ , for  $\delta \in \text{SL}_2(R)$ ,*

$$(\delta \perp I_{2n-2}) E \begin{pmatrix} \lambda x & \lambda y \\ \mu x & \mu y \end{pmatrix} (\delta \perp I_{2n-2})^{-1} = E \begin{pmatrix} \lambda' x & \lambda' y \\ \mu' x & \mu' y \end{pmatrix},$$

for some  $\lambda', \mu' \in R$ .

*Proof.* Take  $\lambda' = \lambda a + \mu b$ ,  $\mu' = c\lambda + \mu d$ , where  $e_1\delta = (a, b)$ ,  $e_2\delta = (c, d)$ .  $\square$

We now come to the Initial Structure Theorem:

**Theorem 2.10. (Initial Structure Theorem:)**

*Let  $R$  be a commutative ring. Then  $\text{ESp}_{2n}(R)$  is generated by matrices of type  $E_2(R)$ ,  $E(A)$ ,  $E(B)$ ,  $E(C)$ ,  $E(D)$ ,  $I_2 + C$ ,  $I_2 + B$ , the last two being “suitably placed” elementary symplectic matrices.*

*Furthermore, one has a decomposition of a  $\varepsilon_\psi \in \text{ESp}_{2n}(R)$  as a product of the type,*

$$\begin{aligned} \varepsilon_\psi &= \delta \begin{pmatrix} A+C \\ D+C \end{pmatrix} \begin{pmatrix} B+D \\ B+A \end{pmatrix} \cdots \begin{pmatrix} B+D \\ B+A \end{pmatrix} \\ &= \delta \begin{pmatrix} A+C \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ D+C \end{pmatrix} \begin{pmatrix} B+D \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ B+A \end{pmatrix} \cdots \begin{pmatrix} A+C \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ D+C \end{pmatrix} \\ &= {}^\delta E(A)E(C)_{I_2+C}E(B)E(D)_{I_2+B} \cdots E(A)E(C)_{I_2+C}E(B)E(D)_{I_2+B} \\ &= (\delta_1 \perp \cdots \perp \delta_n) E(A)E(C)E(B)E(D) \cdots E(A)E(C)E(B)E(D) \\ &= {}^\delta E(A)E(C)E(B)E(D) \cdots E(A)E(C)E(B)E(D) \end{aligned} \quad \begin{array}{l} \text{(A)} \\ \text{(B)} \\ \text{(C)} \\ \text{(D)} \\ \text{(E)} \end{array}$$

for some  $\delta, \delta_1, \dots, \delta_n \in E_2(R)$

Proof: Via standard commutator relations one can show that  $\text{ESp}_{2n}(R)$  is generated by means of the elementary symplectic generators  $E_{21}(x)$ ,  $E_{12}(x)$ ,  $S_{1i}(x)$ ,  $S_{2j}(x)$ , for  $x \in R$ ,  $3 \leq i, j \leq 2n$ .

One has a similar identity to that of Lemma 2.6 on conjugating an expression of type  $\prod_{j=3}^{2n} S_{2j}(y_j)$ . Therefore, via Lemma 2.9, if  $\varepsilon_\psi \in \text{ESp}_{2n}(R)$ , then  $\varepsilon_\psi$  can be written as a product of the form,

$$\varepsilon_\psi =^\delta \prod E \begin{pmatrix} \lambda x & \lambda y \\ \mu x & \mu y \end{pmatrix}, \quad (4)$$

for some  $\delta \in E_2(R)$ ,  $x$ 's,  $y$ 's,  $\lambda$ 's,  $\mu$ 's in  $R$ . This is got by “moving” all the  $E_{21}(\ast)$ 's and  $E_{12}(\ast)$ 's occuring to the “left”. (In fact, one can also assert that each  $(\lambda, \mu) \sim_E (1, 0)$ .)

In the notation of Lemma 2.7

$$E \begin{pmatrix} \lambda x & \lambda y \\ \mu x & \mu y \end{pmatrix} =^{Ch} E \begin{pmatrix} \lambda a & \lambda a \\ \mu a & \mu a \end{pmatrix} E \begin{pmatrix} \lambda b & -\lambda b \\ \mu b & -\mu b \end{pmatrix}, \quad (5)$$

where  $Ch$  is the matrix given in Lemma 2.7. Note that  $Ch$  is an elementary matrix in  $E_2(R)$  as  $(\lambda, \mu) \sim_E (1, 0)$ .

Now pooling together the three identitites (1-3) we get an expression, for  $\lambda, \mu, x, y \in R$ ,

$$E \begin{pmatrix} \lambda x & \lambda y \\ \mu x & \mu y \end{pmatrix} =^{Ch} E(A)E(C)_{I_2+C}E(B)E(D)_{I_2+B}. \quad (6)$$

Now Identity (C) follows from Equation (5) via Lemma 2.8, Lemma 2.9.

Proof of Identity (D): We begin with the situation in Identity (C). Using the equations in Lemma 4.1 (see later) we can move the matrices of the type  $\{I_{2s} \perp (I_2 + B) \perp I_{2t}\}$ ,  $2s + 2 + 2t = 2n$ ,  $\{I_{2s} \perp (I_2 + C) \perp I_{2t}\}$ , for some  $s \geq 1$ ,  $t \geq 0$ , to the left. Lemma 4.1 shows that this is possible without changing the form too much. Sometimes there are additional terms input, but this term does not hinder us from taking the matrix to the left. Moreover, the process is finite and terminates with an element of the type  $(\delta_1 \perp \dots \perp \delta_n)$  as required. we may assume that  $\delta_2, \dots, \delta_n$  are products of elements of the form  $I_2 + B$  and  $I_2 + C$ .

We will deduce Identity (E) from Identity (D) by using the commutator relations of type  $[E(X), E(Y)]$ .

**Lemma 2.11.** *We record all the possible commutator relations of type  $[E(X_i)(x), E(Y_j)(y)]$ , where  $X, Y \in \{A, B, C, D\}$  and  $i, j \in \{2, 3, \dots, n\}$ .*

$[E(X_i)(x), E(X_j)(y)] = I_{2n}$ , for all  $i, j$  where  $X \in \{A, B, C, D\}$ .

$$[E(A_i)(x), E(B_j)(y)] = \begin{cases} I_{2n}, & \text{if } i \neq j; \\ \{I_2 + B(4xy)\} \perp I_{2n-2}, & \text{if } i = j. \end{cases}$$

$$[E(A_i)(x), E(C_j)(y)] = \begin{cases} I_{2(i-1)} \perp \{E(C_{j-i+1})(-2xy)\} \perp I_{2(n-j)}, & \text{if } i < j; \\ I_{2(i-1)} \perp \{I_2 + C(-4xy)\} \perp I_{2(n-i)}, & \text{if } i = j; \\ I_{2(j-1)} \perp \{E(C_{i-j+1})(-2xy)\} \perp I_{2(n-i)}, & \text{if } i > j. \end{cases}$$

$$[E(A_i)(x), E(D_j)(y)] = \begin{cases} I_{2(i-1)} \perp \{E(D_{j-i+1})(-2xy)\} \perp I_{2(n-j)}, & \text{if } i < j; \\ I_{2(j-1)} \perp \{E(D_{i-j+1})(-2xy)\} \perp I_{2(n-i)}, & \text{if } i > j. \end{cases}$$

$$[E(B_i)(x), E(A_j)(y)] = \begin{cases} I_{2n}, & \text{if } i \neq j; \\ \{I_2 + B(-4xy)\} \perp I_{2n-2}, & \text{if } i = j. \end{cases}$$

$$[E(B_i)(x), E(C_j)(y)] = \begin{cases} I_{2(i-1)} \perp \{E(A_{j-i+1})(2xy)\} \perp I_{2(n-j)}, & \text{if } i < j; \\ I_{2(j-1)} \perp \{E(A_{i-j+1})(2xy)\} \perp I_{2(n-i)}, & \text{if } i > j. \end{cases}$$

$$[E(B_i)(x), E(D_j)(y)] = \begin{cases} I_{2(i-1)} \perp \{E(B_{j-i+1})(-2xy)\} \perp I_{2(n-j)}, & \text{if } i < j; \\ I_{2(i-1)} \perp \{I_2 + B(-4xy)\} \perp I_{2(n-i)}, & \text{if } i = j; \\ I_{2(j-1)} \perp \{E(B_{i-j+1})(-2xy)\} \perp I_{2(n-i)}, & \text{if } i > j. \end{cases}$$

$$[E(C_i)(x), E(A_j)(y)] = \begin{cases} I_{2(i-1)} \perp \{E(C_{j-i+1})(2xy)\} \perp I_{2(n-j)}, & \text{if } i < j; \\ I_{2(i-1)} \perp \{I_2 + C(4xy)\} \perp I_{2(n-i)}, & \text{if } i = j; \\ I_{2(j-1)} \perp \{E(C_{i-j+1})(2xy)\} \perp I_{2(n-i)}, & \text{if } i > j. \end{cases}$$

$$[E(C_i)(x), E(B_j)(y)] = \begin{cases} I_{2(i-1)} \perp \{E(D_{j-i+1})(-2xy)\} \perp I_{2(n-j)}, & \text{if } i < j; \\ I_{2(j-1)} \perp \{E(D_{i-j+1})(-2xy)\} \perp I_{2(n-i)}, & \text{if } i > j. \end{cases}$$

$$[E(C_i)(x), E(D_j)(y)] = \begin{cases} I_{2n}, & \text{if } i \neq j; \\ \{I_2 + C(4xy)\} \perp I_{2n-2}, & \text{if } i = j. \end{cases}$$

$$[E(D_i)(x), E(A_j)(y)] = \begin{cases} I_{2(i-1)} \perp \{E(A_{j-i+1})(2xy)\} \perp I_{2(n-j)}, & \text{if } i < j; \\ I_{2(j-1)} \perp \{E(A_{i-j+1})(2xy)\} \perp I_{2(n-i)}, & \text{if } i > j. \end{cases}$$

$$[E(D_i)(x), E(B_j)(y)] = \begin{cases} I_{2(i-1)} \perp \{E(B_{j-i+1})(2xy)\} \perp I_{2(n-j)}, & \text{if } i < j; \\ I_{2(i-1)} \perp \{I_2 + B(4xy)\} \perp I_{2(n-i)}, & \text{if } i = j; \\ I_{2(j-1)} \perp \{E(B_{i-j+1})(2xy)\} \perp I_{2(n-i)}, & \text{if } i > j. \end{cases}$$

$$[E(D_i)(x), E(C_j)(y)] = \begin{cases} I_{2n}, & \text{if } i \neq j; \\ \{I_2 + C(-4xy)\} \perp I_{2n-2}, & \text{if } i = j. \end{cases}$$

$$\begin{aligned}
& [E(A_2)(x), E(D_2)(y)] = \\
& \quad \begin{pmatrix} I_2 + A(8x^2y^2) + A(2xy) + D(2xy) & D(4xy^2) + A(-4x^2y) \\ A(4xy^2) + D(-4x^2y) & I_2 + D(-8x^2y^2) + D(-2xy) + A(-2xy) \end{pmatrix} \\
& [E(B_2)(x), E(C_2)(y)] = \\
& \quad \begin{pmatrix} I_2 + A(2xy) + D(2xy) + A(8x^2y^2) & B(-4x^2y) + C(4xy^2) \\ B(-4x^2y) + C(4xy^2) & I_2 + A(2xy) + D(2xy) + A(8x^2y^2) \end{pmatrix} \\
& [E(C_2)(x), E(B_2)(y)] = \\
& \quad \begin{pmatrix} I_2 + A(-2xy) + D(-2xy) + D(-8x^2y^2) & B(4xy^2) + C(-4x^2y) \\ B(4xy^2) + C(-4x^2y) & I_2 + A(-2xy) + D(-2xy) + D(-8x^2y^2) \end{pmatrix} \\
& [E(D_2)(x), E(A_2)(y)] = \\
& \quad \begin{pmatrix} I_2 + A(-2xy) + D(-2xy) + D(-8x^2y^2) & A(4xy^2) + D(-4x^2y) \\ D(4xy^2) + A(-4x^2y) & I_2 + A(2xy) + D(2xy) + A(8x^2y^2) \end{pmatrix}
\end{aligned}$$

**Corollary 2.12.** *The elements  $\{I_{2r} \perp (I_2 + Y) \perp I_{2s}\}$ ,  $r \geq 0$ ,  $s \geq 0$ ,  $2r + 2 + 2s = 2n$ ,  $Y = B$  or  $C$ , are in the subgroup of  $\text{ESp}_{2n}(R)$  generated by elements of the form  $E(A)$ ,  $E(B)$ ,  $E(C)$ ,  $E(D)$ .*

**Proof:** The relations in Lemma 2.11 show that the “smaller” sized matrices are in the required subgroup.  $\square$

A matrix of type  $(\delta_1 \perp \dots \perp \delta_n)$ , as in Identity (D) can be written as a product of type  $(\delta_1 \perp I_{2n-2})H$ .

Hence, Identity (E) follows from Identity (D).  $\square$

L.N. Vaserstein showed in [12] that the first row of an elementary matrix of even size is the first row of an elementary symplectic matrix, i.e.  $e_{2n}E_{2n}(R) = e_{2n}\text{ESp}_{2n}(R)$ . In view of this, as a consequence of the initial Structure theorem we get:

**Corollary 2.13.** *Let  $R$  be a commutative ring with 1. Then for elementary matrix  $\varepsilon$ , we have  $e_{2n}\varepsilon = e_{2n}\alpha$ , for some elementary symplectic  $\alpha$  in the subgroup  $H$  of  $\text{ESp}_{2n}(R)$ .*

*Proof.* In view of Corollary 2.12 the initial Structure Theorem asserts that if  $\varepsilon_\psi \in \text{ESp}_{2n}(R)$  then  $\varepsilon_\psi =^\delta h$ , for some  $\delta \in E_2(R)$ ,  $h \in H$ . By Vaserstein’s lemma ([12], Lemma 5.6)  $e_{2n}\varepsilon = e_{2n}\varepsilon_\psi$ , for some  $\varepsilon_\psi \in \text{ESp}_{2n}(R)$ . Hence  $e_{2n}\varepsilon = e_{2n}(^\delta h) = e_{2n}\alpha$ , for some  $\alpha \in H$ , as required.  $\square$

**Lemma 2.14.** *Let  $\delta = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$  with  $\det \delta = 1$ . Let  $i \in \{2, 3, \dots, n\}$ . Then we have the following identities:*

$$\begin{aligned}
{}^\delta E(A_i)(x)^{\delta^{-1}} &= E^i \begin{pmatrix} px & px \\ rx & rx \end{pmatrix} E^i \begin{pmatrix} qx & qx \\ sx & sx \end{pmatrix} \{I_{2i-2} \perp \{I_2 + C(-x^2)\} \perp I_{2n-2i}\}. \\
{}^\delta E(B_i)(x)^{\delta^{-1}} &= E^i \begin{pmatrix} px & -px \\ rx & -rx \end{pmatrix} E^i \begin{pmatrix} qx & -qx \\ sx & -sx \end{pmatrix} \{I_{2i-2} \perp \{I_2 + B(x^2)\} \perp I_{2n-2i}\}. \\
{}^\delta E(C_i)(x)^{\delta^{-1}} &= E^i \begin{pmatrix} px & px \\ rx & rx \end{pmatrix} E^i \begin{pmatrix} -qx & -qx \\ -sx & -sx \end{pmatrix} \{I_{2i-2} \perp \{I_2 + C(x^2)\} \perp I_{2n-2i}\}. \\
{}^\delta E(D_i)(x)^{\delta^{-1}} &= E^i \begin{pmatrix} -px & px \\ -rx & rx \end{pmatrix} E^i \begin{pmatrix} qx & -qx \\ sx & -sx \end{pmatrix} \{I_{2i-2} \perp \{I_2 + B(-x^2)\} \perp I_{2n-2i}\}.
\end{aligned}$$



**Corollary 2.15.** *Let  $\delta \in \mathrm{SL}_2(R)$ ,  $\varepsilon \in H(R)$ , the subgroup of  $\mathrm{ESp}_{2n}(R)$ . Then  ${}^\delta \varepsilon {}^{\delta^{-1}} \in H$ .*

*Proof.* This is clear from Lemma 2.14, Equations (2)-(3), and Corollary 2.12.  $\square$

**Corollary 2.16.** *Let  $\delta \in \mathrm{SL}_2(R)$ ,  $\varepsilon_\psi \in \mathrm{ESp}_{2n}(R)$ . Then  ${}^\delta \varepsilon_\psi {}^{\delta^{-1}} = {}^\varepsilon \alpha$ , for some  $\alpha \in H$ ,  $\varepsilon \in \mathrm{E}_2(R)$ .*

*Proof.* This follows from Lemma 2.6, Equations (1)-(3), Corollary 2.12, and Lemma 2.14.  $\square$

### 3 The final Structure Theorem

Our final Structure Theorem is to assert that  $\mathrm{ESp}_{2n}(R)$ ,  $n \geq 2$ , is generated by elements of the type  $\mathrm{E}(A)$ ,  $\mathrm{E}(B)$ ,  $\mathrm{E}(C)$ ,  $\mathrm{E}(D)$ .

Before we come to the final Structure Theorem we make a simple observation:

**Corollary 3.1.** *The subgroup  $\mathrm{E}_2(R) \perp I_{2n-2}$  is contained in  $H$ .*

*Proof.* Let  $\gamma = I_2 + B(c)$ . Then  $\mathrm{E}_{12}(-1)\gamma\mathrm{E}_{12}(1) = \mathrm{E}_{21}(c)$ . By Corollary 2.15  $\mathrm{E}_{12}(-1)\gamma\mathrm{E}_{12}(1) \in H$ , hence  $\mathrm{E}_{21}(c) \in H$ . Similarly, one can show that  $\mathrm{E}_{12}(c) \in H$ .  $\square$

We now come to the **main Structure Theorem for the elementary symplectic group** of size atleast four:

**Theorem 3.2.** *For  $n \geq 2$ ,  $\mathrm{ESp}_{2n}(R)$  coincides with the subgroup  $H$ .*

*Proof.* By the initial Structure Theorem Identity (E) it follows that if  $\varepsilon_\psi \in \mathrm{ESp}_{2n}(R)$  then  $\varepsilon_\psi = {}^\delta h$ , for some  $\delta \in \mathrm{E}_2(R)$ ,  $h \in H$ .

Therefore, it suffices to show that  $(\delta \perp I_{2n-2}) \in H$ , for  $\delta \in \mathrm{E}_2(R)$ . This was shown above in Corollary 3.1.  $\square$

### 4 Local Global Principle for the $A, B, C, D$ subgroup

In this section we give an alternate proof of,  $\mathrm{ESp}_{2n}(R)$  is a normal subgroup of  $\mathrm{Sp}_{2n}(R)$ , from that of V.I. Kopeiko in [8] and G. Taddei in [14]. This proof will throw more light on the commutator relations between the special generators of type  $A, B, C, D$  described above; which we feel is useful to record here.

A sketch of the proof: By Theorem 3.2 we have  $H = \mathrm{ESp}_{2n}(R)$  for  $n \geq 2$ . We prove that  $H$  is a normal subgroup of  $\mathrm{Sp}_{2n}(R)$ ,  $n \geq 2$ . Our idea to prove this is to establish that  $H(R[X])$  satisfies the Local Global principle enunciated by D. Quillen in [10] to settle the Serre's problem on projective modules over a polynomial ring. Our treatment to establish this Local Global principle is influenced by A. Suslin's treatment in [13], which in turn was inspired by D. Quillen's approach in [10]. The treatment of V.I. Kopeiko in [8] is also inspired by [13]; however our treatment is via commutator laws (and not the special forms as in [13], [8]) and is similar to the treatment in [1] for the relative groups via commutator relations.

**Lemma 4.1.** *We record the commutator relations of type  $[E(X_i), \{I_{2j-2} \perp \{I_2 + Y\} \perp I_{2n-2j}\}]$ , with  $X = A, B, C, D$  and  $Y = B, C$  where  $i \in \{2, 3, \dots, n\}$  and  $j \in \{1, 2, \dots, n\}$ .*

$$\begin{aligned} [E(B_i)(x), \{I_{2j-2} \perp \{I_2 + B(y)\} \perp I_{2n-2j}\}] &= I_{2n}, \text{ for all } i, j. \\ [E(C_i)(x), \{I_{2j-2} \perp \{I_2 + C(y)\} \perp I_{2n-2j}\}] &= I_{2n}, \text{ for all } i, j. \end{aligned}$$

$$\begin{aligned} [E(A_i)(x), \{I_{2j-2} \perp \{I_2 + B(y)\} \perp I_{2n-2j}\}] &= \\ &\begin{cases} \{\{I_2 + B(4x^2y)\} \perp I_{2n-2i}\}E(B_i)(2xy), & \text{if } i = j; \\ I_{2n}, & \text{if } i \neq j. \end{cases} \end{aligned}$$

$$\begin{aligned} [E(A_i)(x), \{I_{2j-2} \perp \{I_2 + C(y)\} \perp I_{2n-2j}\}] &= \\ &\begin{cases} \{I_{(2i-2)} \perp \{I_2 + C(4x^2y)\} \perp I_{2n-2i}\}E(C_i)(-2xy), & \text{if } j = 1; \\ I_{2n}, & \text{for any } i, j \text{ with } j \neq 1. \end{cases} \end{aligned}$$

$$\begin{aligned} [E(B_i)(x), \{I_{2j-2} \perp \{I_2 + C(y)\} \perp I_{2n-2j}\}] &= \\ &\begin{cases} \{I_{(2i-2)} \perp \{I_2 + B(-4x^2y)\} \perp I_{2n-2i}\}E(D_i)(-2xy), & \text{if } j = 1; \\ \{\{I_2 + B(-4x^2y)\} \perp I_{2n-2}\}E(A_i)(2xy), & \text{if } i = j; \\ I_{2n}, & \text{if } i \neq j, j \neq 1. \end{cases} \end{aligned}$$

$$\begin{aligned} [E(C_i)(x), \{I_{2j-2} \perp \{I_2 + B(y)\} \perp I_{2n-2j}\}] &= \\ &\begin{cases} \{I_{(2i-2)} \perp \{I_2 + C(-4x^2y)\} \perp I_{2n-2i}\}E(A_i)(-2xy), & \text{if } j = 1; \\ \{\{I_2 + C(-4x^2y)\} \perp I_{2n-2}\}E(D_i)(2xy), & \text{if } i = j; \\ I_{2n}, & \text{if } i \neq j, j \neq 1. \end{cases} \end{aligned}$$

$$\begin{aligned} [E(D_i)(x), \{I_{2j-2} \perp \{I_2 + B(y)\} \perp I_{2n-2j}\}] &= \\ &\begin{cases} \{I_{(2i-2)} \perp \{I_2 + B(4x^2y)\} \perp I_{2n-2i}\}E(B_i)(-2xy), & \text{if } j = 1; \\ I_{2n}, & \text{if } j \neq 1. \end{cases} \end{aligned}$$

$$\begin{aligned} [E(D_i)(x), \{I_{2j-2} \perp \{I_2 + C(y)\} \perp I_{2n-2j}\}] &= \\ &\begin{cases} \{I_{(2i-2)} \perp \{I_2 + B(4x^2y)\} \perp I_{2n-2i}\}E(C_i)(-2xy), & \text{if } j = 1; \\ I_{2n}, & \text{if } j \neq 1. \end{cases} \end{aligned}$$

**Lemma 4.2.** *We record the commutator relations of type  $[E(X_i)(x), E(Y_i)(y)]$ , with  $(X, Y) \in \{(A, D), (B, C)\}$ , where  $i \in \{2, 3, \dots, n\}$ .*

$$\begin{aligned} &[E(A_i)(x), E(D_i)(2yz)] \\ &= [E(A_i)(x), \{\{I_2 + C(4y^2z)\} \perp I_{2n-2}\} \{\{I_2 + C(4y^2z)\} \perp I_{2n-2}\} \\ &\quad [E(A_i)(x), E(C_i)(y)]E(C_i)(y), [E(A_i)(x), \{I_{2i-2} \perp \{I_2 + B(z)\} \perp I_{2n-2i}\}] \\ &\quad \{I_{2i-2} \perp \{I_2 + B(z)\} \perp I_{2n-2i}\}] \\ &\quad [\{I_{2i-2} \perp \{I_2 + B(z)\} \perp I_{2n-2i}\}, E(C_i)(y)]\{\{I_2 + C(-4y^2z)\} \perp I_{2n-2}\}. \end{aligned}$$

$$\begin{aligned}
& [E(B_i)(x), E(C_i)(2yz)] \\
& = [E(B_i)(x), \{I_{(2i-2)} \perp \{I_2 + C(4y^2z)\} \perp I_{2n-2i}\}] \\
& \quad \{I_{(2i-2)} \perp \{I_2 + C(4y^2z)\} \perp I_{2n-2i}\} [E(B_i)(x), E(A_i)(y)] E(A_i)(y), \\
& \quad [E(B_i)(x), \{I_2 + C(-z)\} \perp I_{2n-2}\} \{ \{I_2 + C(-z)\} \perp I_{2n-2}\} ] \\
& \quad [ \{I_2 + C(-z)\} \perp I_{2n-2}\}, E(A_i)(y) ] \{I_{(2i-2)} \perp \{I_2 + C(-4y^2z)\} \perp I_{2n-2i}\}.
\end{aligned}$$

*Proof.* Since

$$\begin{aligned}
E(D_i)(2yz) &= \{ \{I_2 + C(-4y^2z)\} \perp I_{2n-2}\}^{-1} [E(C_i)(y), \{I_{2i-2} \perp \\
& \quad \{I_2 + B(z)\} \perp I_{2n-2i}\}] \text{ and} \\
E(C_i)(2yz) &= \{I_{(2i-2)} \perp \{I_2 + C(-4y^2z)\} \perp I_{2n-2i}\}^{-1} [E(A_i)(y), \\
& \quad \{ \{I_2 + C(-z)\} \perp I_{2n-2}\} ].
\end{aligned}$$

Now using the commutator formula

$$[x, y[z, w]] = [x, y] y [ [x, z]z, [x, w]w ] [w, z] y^{-1},$$

results follows.  $\square$

**Proposition 4.3.** *Let  $X, Y, Z \in \{A, B, C, D\}$ . Let  $k \in \mathbb{N}$  be fixed. Let  $s$  be a non nilpotent element of  $R$ . Let  $m > k$ . Let  $i, j, r_t \in \{2, 3, \dots, n\}$  for every  $t \in \mathbb{N}$  then there exists a product decomposition in  $\text{ESp}_{2n}(R_s)$*

$$E(X_i)(a/s^k)E(Y_j)(s^m x)E(X_i)(a/s^k)^{-1} = \prod_{t=1}^{\lambda} E(Z_{r_t})(s^{m_t} x_t).$$

where  $a$  and  $x$  are elements of  $R$  and the  $x_t$  are suitable elements in  $R$ , also  $\lambda \leq 45$ , with  $m_t \rightarrow \infty$  as  $m \rightarrow \infty$ .

*Proof.* In order to cover all possibilities, we discuss the proof by considering three cases as follows:

**Case 1:** When  $X \in \{A, B, C, D\}$ ,

$$E(X_i)(a/s^k)E(X_i)(s^m x)E(X_i)(a/s^k)^{-1} = E(X_i)(s^m x).$$

For instance,  $E(A_i)(a/s^k)E(A_i)(s^m x)E(A_i)(a/s^k)^{-1} = E(A_i)(s^m x)$ .

**Case 2:** When  $(X, Y) \in \{(A, B), (A, C), (B, A), (B, D), (C, A), (C, D), (D, B), (D, C)\}$

$$E(X_i)(a/s^k)E(Y_j)(s^m x)E(X_i)(a/s^k)^{-1} = \prod_{t=1}^{\lambda} E(Z_{r_t})(s^{m_t} x_t), \text{ where } \lambda \leq 5$$

For instance,

$$\begin{aligned}
E(A_i)(a/s^k)E(B_j)(s^m x)E(A_i)(a/s^k)^{-1} &= [E(A_i)(a/s^k), E(B_j)(s^m x)] E(B_j)(s^m x) \\
&= [E(A_i)(as^p), E(B_j)(s^q x)] E(B_j)(s^m x) \\
&= \prod_{t=1}^{\lambda} E(Z_{r_t})(s^{m_t} x_t) \text{ for } m_t > 0.
\end{aligned}$$

where  $\lambda = 1$ , if  $i \neq j$  and  $\lambda = 5$ , if  $i = j$  (The penultimate equation is via Lemma 2.11, and holds for any positive integers  $p, q$ , with  $p + q = m - k$ .)

$$\begin{aligned} E(A_i)(a/s^k)E(C_j)(s^m x)E(A_i)(a/s^k)^{-1} &= [E(A_i)(a/s^k), E(C_j)(s^m x)]E(C_j)(s^m x) \\ &= [E(A_i)(as^p), E(C_j)(xs^q)]E(C_j)(xs^m) \\ &= \prod_{t=1}^5 E(Z_{r_t})(s^{m_t} x_t) \text{ for } m_t > 0. \end{aligned}$$

(The penultimate equation holds by Lemma 2.11 for any positive integers  $p, q$ , with  $p + q = m - k$ .)

**Case 3(a):** When  $(X, Y) \in \{(A, D), (B, C), (D, A), (C, B)\}$  and  $i \neq j$ , then

$$E(X_i)(a/s^k)E(Y_j)(s^m x)E(X_i)(a/s^k)^{-1} = \prod_{t=1}^5 E(Z_{r_t})(s^{m_t} x_t).$$

We work out the case  $(X, Y) = (A, D)$  below. The other cases can be worked out similarly. For instance,

$$\begin{aligned} E(A_i)(a/s^k)E(D_j)(s^m x)E(A_i)(a/s^k)^{-1} &= [E(A_i)(a/s^k), E(D_j)(s^m x)]E(D_j)(s^m x) \\ &= [E(A_i)(as^p), E(D_j)(s^q x)]E(D_j)(s^m x) \\ &= \prod_{t=1}^5 E(Z_{r_t})(s^{m_t} x_t) \text{ for } m_t > 0. \end{aligned}$$

**Case 3(b):** When  $(X, Y) \in \{(A, D), (B, C), (D, A), (C, B)\}$  and  $i = j$ , then

$$E(X_i)(a/s^k)E(Y_j)(s^m x)E(X_i)(a/s^k)^{-1} = \prod_{t=1}^{\lambda} E(Z_{r_t})(s^{m_t} x_t), \text{ where } \lambda \leq 45.$$

We work out the case  $(X, Y) = (A, D)$  below. The other cases can be worked out similarly.

$$\begin{aligned} &[E(A_i)(x), E(D_i)(2yz)]E(D_i)(2yz) \\ &= [E(A_i)(x), \{\{I_2 + C(4y^2 z)\} \perp I_{2n-2}\}] \{\{I_2 + C(4y^2 z)\} \perp I_{2n-2}\} \\ &\quad [ [E(A_i)(x), E(C_i)(y)]E(C_i)(y), [E(A_i)(x), \{I_{2i-2} \perp \{I_2 + B(z)\} \perp I_{2n-2i}\}] \\ &\quad \{I_{2i-2} \perp \{I_2 + B(z)\} \perp I_{2n-2i}\}][\{I_{2i-2} \perp \{I_2 + B(z)\} \perp I_{2n-2i}\}, E(C_i)(y)] \\ &\quad \{\{I_2 + C(-4y^2 z)\} \perp I_{2n-2}\} E(D_i)(2yz). \end{aligned}$$

We can write it as

$$\begin{aligned} &[E(A_i)(x), E(D_i)(2yz)]E(D_i)(2yz) \\ &= [E(A_i)(-4xz), E(C_i)(xy^2)]E(C_i)(-8xy^2 z) [E(C_i)(y^2), E(D_i)(z)] \\ &\quad [ [E(A_i)(x), E(C_i)(y)]E(C_i)(y), [E(A_i)(x^2), E(B_i)(z)]E(B_i)(2xz) ] \\ &\quad \{I_{2i-2} \perp \{I_2 + B(z)\} \perp I_{2n-2i}\}][E(C_i)(y), E(D_i)(z)]E(D_i)(2yz) \\ &\quad [E(C_i)(y^2), E(D_i)(z)]E(D_i)(2yz). \end{aligned}$$

Now choose  $x = a/s^k, y = s^m$  and  $z = 4us^{2m}$  where  $u \in R$  and  $\{I_{2i-2} \perp \{I_2 + B(z)\} \perp I_{2n-2i}\} = [E(D_i)(us^m), E(B_i)(s^m)]$  then  $xz = 4aus^{2m-k}, xy = as^{m-k}, yz = 4us^{3m}$  and  $xy^2z = 4aus^{4m-k}$ . Now we observe that, for some  $x_t \in R$

$$[E(A_i)(a/s^k), E(D_i)(8us^{3m})] E(D_i)(8us^{3m}) = \prod_{t=1}^{\lambda} E(Z_{r_t})(s^{m_t} x_t).$$

where  $m_t$  is a positive integer such that  $m_t \rightarrow \infty$  as  $m \rightarrow \infty$ .  $\square$

We record a well-known useful observation in the form of a lemma:

**Lemma 4.4.** *Let  $G$  be a group and  $a_i, b_i \in G$ , for  $i = 1, \dots, n$ . Then*

$$\prod_{i=1}^n a_i b_i = \prod_{i=1}^n r_i b_i r_i^{-1} \prod_{i=1}^n a_i,$$

where  $r_i = \prod_{j=1}^i a_j$ .

**Proposition 4.5. (Dilation Principle)**

*Let  $R$  be a commutative ring. Let  $s$  be a non-nilpotent element of  $R$ . Let  $\alpha(X) \in \text{Sp}_{2n}(R[X])$  with  $\alpha(0) = I_{2n}$ .*

*Let  $Y, Z \in \{A, B, C, D\}$ . If  $\alpha_s(X) (= \alpha(X)_s) \in H(R_s[X])$ , then for  $m \gg 0$ , for all  $b \in (s)^m R$ ,  $\alpha(bX) \in H(R[X])$ .*

*Proof.* Let  $\alpha_s(X) = \prod_{k=1}^r E(Y_{i_k})(b_k(X)) \in H(R_s[X])$  where  $b_k(X) \in R_s[X]$  for all  $k$ , also  $i_k \in \{2, 3, \dots, n\}$  for every  $k \in \mathbb{N}$ .

Let  $b_k(X) = b_k(0) + Xb'_k(X)$ . Since  $E(Y)(a+b) = E(Y)(a)E(Y)(b)$ , we can write

$$\alpha_s(X) = \prod_{k=1}^r E(Y_{i_k})(b_k(0))E(Y_{i_k})(Xb'_k(X)).$$

By Lemma 4.4. one has

$$\alpha_s(X) = \prod_{k=1}^r \gamma_k E(Y_{i_k})(Xb'_k(X)) \gamma_k^{-1} \prod_{k=1}^r E(Y_{i_k})(b_k(0)).$$

where  $\gamma_k = \prod_{j=1}^k E(Y_{i_j})(b_j(0))$  and  $i_j \in \{2, 3, \dots, n\}$  for every  $j \in \mathbb{N}$ . As  $\gamma_r = I_{2n}$ , Therefore

$$\alpha_s(X) = \prod_{k=1}^r \gamma_k E(Y_{i_k})(Xb'_k(X)) \gamma_k^{-1}.$$

Hence we can write

$$\alpha_s(s^m X) = \prod_{k=1}^r \gamma_k E(Y_{i_k})(s^m Xb'_k(s^m X)) \gamma_k^{-1}.$$

Our next claim is that if  $\beta = \prod_{j=1}^k E(Y_{i_j})(b_j)$ ,  $b_j \in R_s$  then we can show that one has a product decomposition

$$\beta E(Z)(s^m x) \beta^{-1} = \prod_{t=1}^{\lambda_k} E(Z_{r_t})(s^{m_t} x_t), \quad (7)$$

with  $m_t \rightarrow \infty$  as  $m \rightarrow \infty$ ,  $x_t \in R$ .

We do this by induction on  $k$ . where  $\beta = \beta_1 \beta_2 \cdots \beta_k$ , where  $\beta_j = E(Y_{i_j})(b_j)$ .

When  $k = 1$  by Proposition 4.3 we have a product decomposition

$$\beta_1 E(Z)(s^m x) \beta_1^{-1} = \prod_{t=1}^{\lambda_1} E(Z_{r_t})(s^{m_t} x_t),$$

with  $m_t \rightarrow \infty$  as  $m \rightarrow \infty$ . Now assume the result is true for  $k - 1$ .

$$\beta_1 \beta_2 \cdots \beta_{k-1} E(Z)(s^m x) (\beta_1 \beta_2 \cdots \beta_{k-1})^{-1} = \prod_{t=1}^{\lambda_{k-1}} E(Z_{r_t})(s^{m_t} x_t),$$

Now to show for  $k$ , we have an expression

$$\beta_1 \beta_2 \cdots \beta_{k-1} \beta_k E(Z)(s^m x) (\beta_1 \beta_2 \cdots \beta_{k-1} \beta_k)^{-1}, \quad (8)$$

Since by Proposition 4.3, we can write

$$\beta_k E(Z)(s^m x) \beta_k^{-1} = \prod_{t=1}^{\lambda} E(Z_{r_t})(s^{m_t} x_t) = \theta_1 \theta_2 \cdots \theta_{\lambda} \text{ (say.)}$$

Hence we can write the expression (8) as

$$\beta_1 \beta_2 \cdots \beta_{k-1} \theta_1 \theta_2 \cdots \theta_{\lambda} (\beta_1 \beta_2 \cdots \beta_{k-1})^{-1}$$

Now we rewrite above expression as

$$\begin{aligned} & \beta_1 \beta_2 \cdots \beta_{k-1} \theta_1 (\beta_1 \beta_2 \cdots \beta_{k-1})^{-1} \beta_1 \beta_2 \cdots \beta_{k-1} \theta_2 (\beta_1 \beta_2 \cdots \beta_{k-1})^{-1} \\ & \cdots \beta_1 \beta_2 \cdots \beta_{k-1} \theta_{\lambda} (\beta_1 \beta_2 \cdots \beta_{k-1})^{-1} \end{aligned}$$

Now for each  $\beta_1 \beta_2 \cdots \beta_{k-1} \theta_l (\beta_1 \beta_2 \cdots \beta_{k-1})^{-1}$  as  $l$  varies from 1 to  $\lambda$ , Induction hypothesis for  $k - 1$  can be applied. Hence we have product decomposition as in Equation (7).

Therefore we can write

$$\alpha_s(s^m X) = \prod_{k=1}^r \prod_{t=1}^{\lambda_k} E(Z_{r_t})(s^{m_t} x_t),$$

for  $m$  large enough, the term  $s^{m_t} x_t$  is contained in  $R[X]$ , as required. Hence

$$\alpha(bX) = \prod_{k=1}^r \prod_{t=1}^{\lambda_k} E(Z_{r_t})(s^{m_t} x_t) \in H(R[X]).$$

□

**Theorem 4.6. (Local Global Principle)**

Let  $\alpha(X) \in \text{Sp}_{2n}(R[X])$ , with  $\alpha(0) = I_{2n}$ . If  $\alpha(X)_{\mathfrak{m}} \in H(R_{\mathfrak{m}}[X])$ , for all maximal ideals  $\mathfrak{m}$  of  $R$ , then  $\alpha(X) \in H(R[X])$ .

*Proof.* Let  $\mathfrak{m}$  be a maximal ideal of  $R$ . Choose an element  $a_{\mathfrak{m}}$  from  $R \setminus \mathfrak{m}$  such that  $\alpha(X)_{a_{\mathfrak{m}}} \in H(R_{a_{\mathfrak{m}}}[X])$ . Let us define

$$\beta(X, Y) = \alpha(X + Y)_{a_{\mathfrak{m}}} \alpha(Y)_{a_{\mathfrak{m}}}^{-1}.$$

Clearly  $\beta(X, Y) \in H(R_{a_{\mathfrak{m}}}[X, Y])$ , and  $\beta(0, Y) = I_{2n}$ .

Therefore by Proposition 4.5,  $\beta(b_{\mathfrak{m}}X, Y) \in H(R[X, Y])$ , where  $b_{\mathfrak{m}} \in (a_{\mathfrak{m}}^N)$ , for some  $N \gg 0$ .

The ideal generated by the  $b_{\mathfrak{m}}$  is the whole ring  $R$ . Therefore we have  $c_1 b_{\mathfrak{m}_1} + c_2 b_{\mathfrak{m}_2} + \dots + c_k b_{\mathfrak{m}_k} = 1$ , where  $c_i \in R$ , for  $1 \leq i \leq k$ .

Note that  $\beta(c_i b_{\mathfrak{m}_i} X, Y) \in H(R[X, Y])$ , for  $1 \leq i \leq k$ .

$$\alpha(X) = \left( \prod_{i=1}^{k-1} \beta(c_i b_{\mathfrak{m}_i} X, T_i) \right) \beta(c_k b_{\mathfrak{m}_k} X, 0),$$

where  $T_i = c_{i+1} b_{\mathfrak{m}_{i+1}} X + \dots + c_k b_{\mathfrak{m}_k} X$ . Hence  $\alpha(X) \in H(R[X])$  □

**Corollary 4.7.** *The subgroup  $H(R)$  (viz.  $\mathrm{ESp}_{2n}(R)$  by Theorem 3.2) is a normal subgroup of  $\mathrm{Sp}_{2n}(R)$ .*

*Proof.* Let  $\gamma \in \mathrm{Sp}_{2n}(R)$ ,  $h \in H(R)$ . Choose a homotopy  $h(T) \in H(R[T])$  of  $h$ . Consider  $\gamma h(T) \gamma^{-1}$ .

Note that for a prime ideal  $\mathfrak{p}$  of  $R$ ,  $\mathrm{Sp}_{2n}(R_{\mathfrak{p}}) = \mathrm{ESp}_{2n}(R_{\mathfrak{p}})$ . By Theorem 2.10 (E),  $\gamma_{\mathfrak{p}} = {}^{\delta(\mathfrak{p})} \alpha$ , for some  $\alpha \in H(R_{\mathfrak{p}})$ .

Thus,  $\gamma_{\mathfrak{p}} h(T)_{\mathfrak{p}} \gamma_{\mathfrak{p}}^{-1} = {}^{\delta} \alpha h(T)_{\mathfrak{p}} ({}^{\delta} \alpha)^{-1}$ , for some  $\alpha \in H(R_{\mathfrak{p}})$ ,  $\delta \in E_2(R_{\mathfrak{p}})$ . By Lemma 2.15,  $\gamma_{\mathfrak{p}} h(T)_{\mathfrak{p}} \gamma_{\mathfrak{p}}^{-1} \in H(R_{\mathfrak{p}}[T])$ , for all primes  $\mathfrak{p}$  of  $R$ .

By the Local Global Principle proved in Theorem 4.6

$$\gamma h(T) \gamma^{-1} \in \gamma h(0) \gamma^{-1} H(R[T]) = H(R[T]),$$

as  $h(0) = I_{2n}$ . Hence  $\gamma h(T) \gamma^{-1} \in H(R[T])$ . Hence  $\gamma h(1) \gamma^{-1} \in H(R)$ , as required. □

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